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Levinson's Limit-Point Criterion and Powers*

W. D. EVANS

Department of Pure Mathematics, University College, Cardiff, Wales

AND

A. ZETTL

Department of Mathematics, University of Dundee, Scotland,
and
Department of Mathematical Sciences, Northern Illinois University, De Kalb, Illinois 60115

Submitted by G.-C. Rota

It is shown here that Levinson's well-known criterion for the real differential expression $My = -(py')' + qy$ to be in the limit-point case also implies that M^k is in the limit-point case for any $k = 1, 2, 3,...$ and hence that any polynomial expression in M is in the limit-point case.

1. INTRODUCTION

In his now classic study of the expansion theory in terms of eigenfunctions of singular boundary-value problems Weyl in [16] classified second-order linear differential expressions

$$My = -(py')' + qy \qquad (' \equiv (d/dt)) \tag{1.1}$$

with $p > 0$ and p, q real-valued functions on $[0, \infty)$ into two mutually exclusive classes. Those expressions M such that all solutions of $My = 0$ are in $L^2(0, \infty)$ are said to be in the limit-circle case, the others in the limit-point case. In the limit-point case the equation $My = \lambda y$ has one and only one linearly independent $L^2(0, \infty)$ solution for any nonreal number λ.

Necessary and sufficient conditions on the coefficients for the limit-point (or limit-circle) case are not known. Many sufficient conditions are known. Perhaps the best-known and most widely quoted criterion for the limit-point case is that in the result of Levinson in [12]:

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M is in the limit-point case if there exist a positive differentiable function Q and positive constants k_1, k_2 such that for some number $a > 0$,

$$q(t) \geqslant -k_1 Q(t), \qquad t \geqslant a, \tag{1.2a}$$

$$p(t) Q'^2(t) Q^{-3}(t) \leqslant k_2, \qquad t \geqslant a, \tag{1.2b}$$

$$\int_a^\infty \{p(t) Q(t)\}^{-1/2} \, dt = \infty. \tag{1.2c}$$

In the special case when $p(t) = 1$, we can take $Q(t) = t^2$ in (1.2a) and (1.2b), (1.2c) will hold. In other words if

$$p(t) = 1, \qquad q(t) \geqslant -k_1 t^2, \qquad k_1 > 0, \quad t \geqslant a, \tag{1.3}$$

then (1.1) is in the limit-point case. This condition is best possible in the sense that t^2 cannot be replaced by a higher power of t in (1.3). (See [3, pp. 1409–1410].)

The Weyl theory was extended to higher-order, real, symmetric expressions by Glazman [8] and Kodaira [10]. For

$$Ny = \sum_{j=0}^{n} (p_j y^{(j)})^{(j)} \tag{1.4}$$

with $p_n > 0$ and p_j real, $j = 0, 1, \ldots, n$, the deficiency index d can be defined as the number of linearly independent solutions of the equation

$$Ny = \lambda y, \qquad \operatorname{Im} \lambda \neq 0,$$

which are in $L^2(0, \infty)$. This integer d is independent of λ provided $\operatorname{Im} \lambda \neq 0$ (see [13]). Since d depends only on the expression N itself we will indicate this by writing $d = d(N)$.

According to Glazman's classification result we have

$$n \leqslant d(N) \leqslant 2n$$

and all values of $d(N)$ within this range occur. Following Weyl's terminology in the second-order case we will say that N is in the limit-point case, or N is limit-point if $d(N) = n$ and in the limit-circle case or limit-circle if $d(N) = 2n$. For a discussion of the general theory of ordinary linear differential operators including the elementary facts quoted above the reader is referred to the books by Akhiezer and Glazman [1] and Naimark [13].

In this paper we show that condition (1.2) is sufficient not only for M to be limit-point as Levinson showed but also for M^k (if it is defined) to be limit-point for any $k = 2, 3, 4, \ldots$. Special cases of this result have previously been obtained by Everitt and Giertz [4, 5], Kumar [11], and Read [14]. Everitt and Giertz in [5] show that condition (1.3) implies M^2 is limit-point. (See Corollary 2 below.)

Our proof is based on the property of partial separation for powers M^k of differential expressions. This property has been used by Chaudhuri and Everitt [2] and Everitt and Giertz [4, 5] for powers of second-order expressions M in the limit-point case. The general theory of the relationship between partial separation and the deficiency indices of powers M^k is developed—by different and more direct methods—in [17, 18] for symmetric expressions M of any order with real or complex coefficients. The technique for establishing partial separation is based on the method in [7], where it was used to find limit-point criteria for fourth-order expressions. It is also similar in some respects to the method employed by Everitt and Giertz in [5] but it is different from that of Read [14]. Under our conditions the minimal operator need not have a closed range—this property plays an important part in Read's proof.

2. The Main Results

Powers of the symmetric differential expression M are defined in the natural way:

$$M^2y = M(My), \qquad M^{k+1}y = M(M^ky), \qquad k = 1, 2, 3,\ldots.$$

Similarly polynomial expressions $p(M)$ are defined for any polynomial p. If the polynomial p has real coefficients then $p(M)$ is a symmetric differential expression (see [3]).

We shall always assume that p and q are sufficiently smooth for the powers of M to be defined and to be regular at the end point 0 of the interval $[0, \infty)$. For instance, whenever we consider M^k we assume that the derivatives $p^{(2k-2)}$ and $q^{(2k-3)}$ are in $AC_{\text{loc}}[0, \infty)$, the space of functions which are absolutely continuous on compact subsets of $[0, \infty)$. Also when we write M^kf we shall be assuming that $f^{(2k-1)}$ is in $AC_{\text{loc}}[0, \infty)$, and we denote the set of such functions f by $AC_k[0, \infty)$.

We can now state our main results. In these results k is an arbitrary positive integer.

Theorem 1. *If the Levinson condition* (1.2) *is satisfied then M^k is limit-point.*

As a special case of this we have:

Corollary 2. *Suppose that there exist positive numbers k_1, k_2 and a and a number $\alpha \leqslant 2$ such that, for $t \geqslant a$,*

$$0 < p(t) < k_1 t^{\alpha}, \qquad q(t) > -k_2 t^{2-\alpha}. \tag{2.1}$$

Then M^k is limit-point. In particular, this implies that M^k is limit-point if (1.3) is satisfied.

The corollary follows immediately from the theorem on putting $Q(t) = t^{2-\alpha}$. The case $k = 2$ of Corollary 2 was established by Everitt and Giertz in [5].

In [9] Kauffman shows that M^k is limit-point if and only if $p(M)$ is limit-point for any polynomial p of degree k with real coefficients. We therefore get for $C^\infty[0, \infty)$ coefficients p, q,

COROLLARY 3. *If* (1.2) *is satisfied then $p(M)$ is limit-point for any polynomial p with real coefficients.*

A remarkable feature of the above theorems is that although the polynomials $p(M)$ involve derivatives of p and q no growth conditions of any kind are put on these derivatives.

3. Proof of Theorem 1

The proof of Theorem 1 makes use of the following concept of partial separation of powers.

DEFINITION. We say that M^k is partially separated if

$$f, M^k f \in L^2(0, \infty) \Rightarrow M^j f \in L^2(0, \infty) \qquad \text{for } j = 1, 2, \dots, k-1. \tag{3.1}$$

A consequence of this property which we use to establish Theorem 1 is the following result of Everitt and Giertz in [4] (see also [17, 18]).

THEOREM 4. *Let M be given by* (1.1) *and suppose that M is limit-point. If M^k is partially separated then M^k is limit-point.*

The proof that M^k is partially separated (and hence M^k is limit-point) under condition (1.2) will follow from the following four technical lemmas. The first one is a result proved by Read in [15]. Since Read's proof is somewhat hidden we reproduce it here for the convenience of the reader.

All the integrals appearing below are with respect to Lebesque measure, but in order to simplify things we omit the dx. Also we shall use κ, κ_1, κ_2,..., to denote various absolute positive constants and ϵ, ϵ_1, ϵ_2,... to denote various "small" positive constants, these constants not necessarily being the same on each occurrence. We write $\kappa(\epsilon)$ to indicate dependence on the number ϵ.

LEMMA 5 (Read [15].) *Let M be given by* (1.1) *and suppose that there is some positive differentiable function Q such that* (1.2) *is satisfied. Then there exist a positive differentiable function Q_1 and positive numbers a_1, δ such that* (1.2) *holds with $Q = Q_1$ for $t \geqslant a_1$ and, in addition, $Q_1(t) \geqslant \delta > 0$ for all $t \geqslant a_1$.*

Proof. We shall prove that

$$Q_1(t) = \left(\int_a^t (pQ)^{-1/2}\right)^2 Q(t)$$

satisfies our requirements.

First observe that from (1.2c) there exists a number $a_1 > a$ such that for $t \geqslant a_1$, $Q_1(t) \geqslant Q(t)$ and hence $q(t) \geqslant -k_1 Q_1(t)$ for $t \geqslant a_1$. Also

$$\int_{a_1}^T (pQ_1)^{-1/2} = \log \left\{\int_a^T (pQ)^{-1/2}\right\} - \log \left\{\int_a^{a_1} (pQ)^{-1/2}\right\} \to \infty$$

as $T \to \infty$ and hence (1.2c) is satisfied with $Q = Q_1$ and $a = a_1$.

For the remainder of the lemma we begin with

$$p^{1/2}(Q_1^{-1/2})' = p^{1/2}(Q^{-1/2})' \Big/ \int_a^t (pQ)^{-1/2} - 1/Q_1 .$$

Hence, in view of (1.2b) the proof will be complete if we show that $Q_1 \geqslant \delta > 0$ for some δ and $t \geqslant a_1$.

Let $t_0 > a_1$. If $Q(t_0) \geqslant 1$ then by the above choice of a_1, $Q_1(t_0) \geqslant Q(t_0) \geqslant 1$. If $Q(t_0) < 1$ let

$$s_0 = \max\{a_1 \leqslant t < t_0 \mid Q(t) \geqslant 1\}$$

with the understanding that $s_0 = a_1$ if the above set is empty. Then

$$\begin{aligned} Q^{-1/2}(t_0) - Q^{-1/2}(s_0) &= \int_{s_0}^{t_0} (Q^{-1/2})' \leqslant \tfrac{1}{2}(k_2)^{1/2} \int_{s_0}^{t_0} p^{-1/2} \\ &\leqslant \tfrac{1}{2}(k_2)^{1/2} \int_{s_0}^{t_0} (pQ)^{-1/2} \\ &\leqslant \tfrac{1}{2}(k_2)^{1/2} \int_a^{t_0} (pQ)^{-1/2}. \end{aligned}$$

Hence, since $Q(s_0) \geqslant 1$ if $s_0 > a_1$ ($Q(s_0)$ being constant if $s_0 = a_1$),

$$\begin{aligned} Q_1^{-1/2}(t_0) &\leqslant \left\{Q^{-1/2}(s_0) + \tfrac{1}{2}(k_2)^{1/2} \int_a^{t_0} (pQ)^{-1/2}\right\} \Big/ \int_a^{t_0} (pQ)^{-1/2} \\ &\leqslant \kappa + \tfrac{1}{2}(k_2)^{1/2}. \end{aligned}$$

It therefore follows that $Q_1(t) \geqslant (\kappa + \frac{1}{2}(k_2)^{1/2})^{-2}$ for all $t \geqslant a_1$ and the lemma is proved.

It is clearly enough to establish partial separation for real functions f in (3.1). Hence in the remainder of the paper the functions f appearing will be assumed to be real.

LEMMA 6. *Let Q be a positive, differentiable function and let v be a nonnegative, piecewise continuously differentiable function whose support is a closed interval $I \subset (0, \infty)$. Suppose that, for some positive constants k_i, $i = 1, 2, 3, 4, 5$, independent of I, the following inequalities hold* a.e. *in I:*

(i) $(pQ)^{1/2} v' \leqslant k_1$,

(ii) $q \geqslant -k_2 Q$,

(iii) $pQ'^2 Q^{-3} \leqslant k_3$,

(iv) $|v| \leqslant k_4$, $Q \geqslant k_5$.

Then, given any $\epsilon > 0$ there exists a number $\kappa = \kappa(\epsilon) > 0$, independent of I, such that (with $M^0 f = f$),

$$\int_I p v^{4j-2} Q^{-1} (M^{j-1} f)'^2 \leqslant \epsilon \int_I v^{4j} (M^j f)^2 + \kappa(\epsilon) \int_I v^{4j-4} (M^{j-1} f)^2 \tag{3.2}$$

for $j = 1, 2, \ldots, k$ and any f in $AC_k[0, \infty)$.

Proof. The proof involves integration by parts and the repeated use of the inequality $2|ab| \leqslant \epsilon a^2 + (1/\epsilon) b^2$ which holds for arbitrary $\epsilon > 0$. All the integrals appearing will be over I. In those steps below which involve integration by parts the integration constants vanish since v vanishes at the end points of I.

Let $g = M^{j-1} f$. We get on integration by parts, writing Q^{-1} for $1/Q$,

$$\begin{aligned}
&\int p v^{4j-2} Q^{-1} g'^2 \\
&\quad = -\int \{ v^{4j-2} Q^{-1} (pg')' + (4j-2) v^{4j-3} v' Q^{-1} p g' - v^{4j-2} Q^{-2} Q' p g' \} g \\
&\quad = \int v^{4j-2} Q^{-1} (Mg - qg) g \\
&\qquad - \int (4j-2) Q^{-1} (p^{1/2} v' Q^{1/2}) (p^{1/2} v^{2j-1} Q^{-1/2} g') (v^{2j-2} g) \\
&\qquad + \int (p^{1/2} Q' Q^{-3/2}) (p^{1/2} v^{2j} Q^{-1/2} g') (v^{2j-2} g) \\
&\quad \leqslant \kappa_1 \int v^{4j-2} g M g + \kappa_2 \int v^{4j-4} g^2 + \kappa_3 \int (p^{1/2} v^{2j-1} Q^{-1/2} g') (v^{2j-2} g)
\end{aligned}$$

from (i) to (iv),

$$\begin{aligned}
\int p v^{4j-2} Q^{-1} g'^2 \leqslant \kappa_1 \left\{ \int v^{4j} (Mg)^2 \int v^{4j-4} g^2 \right\}^{1/2} + \kappa_2 \int v^{4j-4} g^2 \\
+ \kappa_3 \left\{ \int p v^{4j-2} Q^{-1} g'^2 \int v^{4j-4} g^2 \right\}^{1/2}
\end{aligned}$$

by the Cauchy–Schwarz inequality,

$$\int pv^{4j-2}Q^{-1}g'^2 \leqslant \tfrac{1}{2}\kappa_1 \left\{\epsilon_1 \int v^{4j}(Mg)^2 + (1/\epsilon_1) \int v^{4j-4}g^2\right\} + \kappa_2 \int v^{4j-4}g^2$$

$$+ \tfrac{1}{2}\kappa_3 \left\{\epsilon_2 \int pv^{4j-2}Q^{-1}g'^2 + (1/\epsilon_2) \int v^{4j-4}g^2\right\},$$

for arbitrary ϵ_1, $\epsilon_2 > 0$. The lemma follows by choosing ϵ_2 sufficiently small.

LEMMA 7. *Under the hypothesis of Lemma* 6, *given any* $\epsilon > 0$ *there exists a positive number* $\kappa = \kappa(\epsilon)$, *independent of* I, *such that*

$$\int_I v^{4j}(M^jf)^2 \leqslant \epsilon \int_I v^{4j+4}(M^{j+1}f)^2 + \kappa(\epsilon) \int_I v^{4j-4}(M^{j-1}f)^2 \tag{3.3}$$

for $j = 1, 2,..., k-1$ *and any* f *in* $AC_k[0, \infty)$.

Proof. We have on integration by parts,

$$\begin{aligned}
&\int v^{4j}M^{j-1}f M^{j+1}f \\
&= \int v^{4j}M^{j-1}f\{-(p[M^jf]')' + qM^jf\} \\
&= \int \{v^{4j}p(M^{j-1}f)'(M^jf)' + 4jv^{4j-1}v'pM^{j-1}f(M^jf)' + v^{4j}qM^{j-1}fM^jf\} \\
&= -4j \int v^{4j-1}v'p(M^{j-1}f)' M^jf + 4j \int v^{4j-1}v'pM^{j-1}f(M^jf)' + \int v^{4j}(M^jf)^2 \\
&= 4jI_1 + 4jI_2 + \int v^{4j}(M^jf)^2, \qquad (3.4)
\end{aligned}$$

say. For any $\epsilon_1 > 0$,

$$\begin{aligned}
|I_1| &= \left| \int (p^{1/2}v'Q^{1/2})(p^{1/2}v^{2j-1}[M^{j-1}f]'Q^{-1/2})(v^{2j}M^jf) \right| \\
&\leqslant \kappa_1 \left\{\int pv^{4j-2}Q^{-1}(M^{j-1}f)'^2\right\}^{1/2} \left\{\int v^{4j}(M^jf)^2\right\}^{1/2}
\end{aligned}$$

by the Cauchy–Schwarz inequality and (i),

$$\begin{aligned}
|I_1| &\leqslant \epsilon_1 \int v^{4j}(M^jf)^2 + (\kappa_2/\epsilon_1) \int pv^{4j-2}Q^{-1}(M^{j-1}f)'^2 \\
&\leqslant \epsilon_1 \int v^{4j}(M^jf)^2 + (\kappa_2/\epsilon_1) \left\{\epsilon_2 \int v^{4j}(M^jf)^2 + \kappa_3(\epsilon_2) \int v^{4j-4}(M^{j-1}f)^2\right\}
\end{aligned}$$

for any $\epsilon_1, \epsilon_2 > 0$, from Lemma 6. Hence, by suitable choice of ϵ_1, ϵ_2 we have for any $\epsilon_3 > 0$,

$$|I_1| \leqslant \epsilon_3 \int v^{4j}(M^j f)^2 + \kappa(\epsilon_3) \int v^{4j-4}(M^{j-1}f)^2. \tag{3.5}$$

Similarly, we have

$$\begin{aligned}
|I_2| &\leqslant \kappa_1 \int (p^{1/2}v^{2j+1}Q^{-1/2}(M^j f)')\,(v^{2j-2}M^{j-1}f) \\
&\leqslant \epsilon_1 \int pv^{4j+2}Q^{-1}(M^j f)'^2 + (\kappa_2/\epsilon_1) \int v^{4j-4}(M^{j-1}f)^2 \\
&\leqslant \epsilon_1 \left\{\epsilon_2 \int v^{4j+4}(M^{j+1}f)^2 + \kappa_3(\epsilon_2) \int v^{4j}(M^j f)^2\right\} + (\kappa_2/\epsilon_1) \int v^{4j-4}(M^{j-1}f)^2
\end{aligned}$$

from Lemma 6. Hence, we can choose ϵ_1, ϵ_2 such that, for any $\epsilon_4 > 0$,

$$|I_2| \leqslant \epsilon_4 \int v^{4j+4}(M^{j+1}f)^2 + \epsilon_4 \int v^{4j}(M^j f)^2 + \kappa(\epsilon_4) \int v^{4j-4}(M^{j-1}f)^2. \tag{3.6}$$

Hence, from (3.4), (3.5), and (3.6)

$$\begin{aligned}
&\int v^{4j}M^{j-1}fM^{j+1}f \\
&\quad\geqslant \int v^{4j}(M^j f)^2 - 4j \left\{\epsilon_3 \int v^{4j}(M^j f)^2 + \kappa(\epsilon_3) \int v^{4j-4}(M^{j-1}f)^2\right\} \\
&\quad\quad - 4j \left\{\epsilon_4 \int v^{4j+4}(M^{j+1}f)^2 + \epsilon_4 \int v^{4j}(M^j f)^2 + \kappa(\epsilon_4) \int v^{4j-4}(M^{j-1}f)^2\right\}.
\end{aligned}$$

It therefore follows on choosing ϵ_3 and ϵ_4 sufficiently small, that

$$\begin{aligned}
&\int v^{4j}(M^j f)^2 \\
&\quad\leqslant \kappa_1 \int v^{4j}M^{j-1}fM^{j+1}f + \kappa_2\epsilon_4 \int v^{4j+4}(M^{j+1}f)^2 + \kappa_3(\epsilon_4) \int v^{4j-4}(M^{j-1}f)^2 \\
&\quad\leqslant \epsilon_5 \int v^{4j+4}(M^{j+1}f)^2 + \kappa_4(\epsilon_5) \int v^{4j-4}(M^{j-1}f)^2.
\end{aligned}$$

This completes the proof of Lemma 7.

LEMMA 8. *Under the hypothesis of Lemma 6, given any $\epsilon > 0$ there exists a positive number $\kappa = \kappa(\epsilon)$, independent of I, such that*

$$\int_I v^{4j}(M^j f)^2 \leqslant \epsilon \int_I v^{4k}(M^k f)^2 + \kappa(\epsilon) \int_I f^2 \tag{3.7}$$

for $j = 1, 2, \ldots, k-1$ and any f in $AC_k(0, \infty)$.

Proof. The proof is by induction on k. The case $k = 2$ is given by Lemma 7. Let us assume that (3.7) holds for all integers up to k. We wish to prove that if s is an integer satisfying $1 \leqslant s \leqslant k$ then given $\epsilon > 0$ there exists a $\kappa = \kappa(\epsilon) > 0$ such that, for any $f \in AC_{k+1}(0, \infty)$,

$$\int v^{4s}(M^s f)^2 \leqslant \epsilon \int v^{4k+4}(M^{k+1}f)^2 + \kappa(\epsilon) \int f^2. \tag{3.8}$$

By Lemma 7 and the induction hypothesis

$$\int v^{4k}(M^k f)^2 \leqslant \epsilon_1 \int v^{4k+4}(M^{k+1}f)^2 + \kappa(\epsilon_1) \int v^{4k-4}(M^{k-1}f)^2$$

$$\leqslant \epsilon_1 \int v^{4k+4}(M^{k+1}f)^2 + \kappa(\epsilon_1) \left\{\epsilon_2 \int v^{4k}(M^k f)^2 + \kappa(\epsilon_2) \int f^2\right\}.$$

By a suitable choice of ϵ_1 and ϵ_2 (3.8) follows when $s = k$. If $s < k$, we again get by the induction hypothesis

$$\int v^{4s}(M^s f)^2 \leqslant \epsilon_1 \int v^{4k}(M^k f)^2 + \kappa(\epsilon_1) \int f^2$$

$$\leqslant \epsilon_1 \left\{\epsilon_2 \int v^{4k+4}(M^{k+1}f)^2 + \kappa(\epsilon_2) \int f^2\right\} + \kappa(\epsilon_1) \int f^2$$

from above (i.e., (3.8) with $s = k$)

$$\int v^{4s}(M^s f)^2 \leqslant \epsilon \int v^{4k+4}(M^{k+1}f)^2 + \kappa(\epsilon) \int f^2$$

by choice of ϵ_1 and ϵ_2. This completes the proof of Lemma 8.

We can now prove Theorem 1.

Proof of Theorem 1. From Lemma 5 we can assume that in (1.2), $Q(t) \geqslant \delta > 0$ for $t \geqslant a$.

Let

$$\theta(t) = \int_a^t (pQ)^{-1/2}, \qquad t > a,$$

so that θ is strictly increasing and $\theta(t) \to \infty$ as $t \to \infty$. For $T > a + 1$ define

$$\begin{aligned} v(t) &= 1 - \exp[\theta(t) - \theta(T)] \qquad && \text{for } a + 1 < t < T \\ &= 0 && \text{for } t \geqslant T, \end{aligned} \tag{3.9}$$

and in $[a, a + 1]$ choose v such that v vanishes in a right neighborhood of a, $0 \leqslant v(t) \leqslant 1$ and $|v'(t)| \leqslant 1$, $a \leqslant t \leqslant a + 1$. Then $(pQ)^{1/2} v'$ is bounded in $[a, a + 1]$ and, for $a + 1 < t < T$,

$$|(pQ)^{1/2} v'| = \exp(\theta(t) - \theta(T)) \leqslant 1.$$

Hence condition (i) of Lemma 6 is satisfied. The other conditions in Lemma 6 are also satisfied and so we have from (3.7) with $I = [a, T]$, for $j = 1, 2, \ldots, k - 1$,

$$\int_a^T v^{4j}(M^j f)^2 \leqslant \kappa_1 \int_0^\infty (M^k f)^2 + \kappa_2 \int_0^\infty f^2 \tag{3.10}$$

if $f, M^k f \in L^2(0, \infty)$.

Now choose T and X such that

$$\theta(T) - \log 2 > 0 \qquad \text{and} \qquad \theta(X) = \theta(T) - \log 2$$

so that $X \to \infty$ as $T \to \infty$. For $a \leqslant t \leqslant X$, $\theta(t) \leqslant \theta(X)$ and hence $1 - \exp[\theta(t) - \theta(T)] \geqslant \frac{1}{2}$. It therefore follows from (3.9) and (3.10) that

$$\begin{aligned}\int_{a+1}^X (\tfrac{1}{2})^{4j}\, (M^j f)^2 &\leqslant \int_a^T v^{4j}(M^j f)^2 \\ &\leqslant \kappa_1 \int_0^\infty (M^k f)^2 + \kappa_2 \int_0^\infty f^2.\end{aligned}$$

We have therefore shown that $M^j f \in L^2(0, \infty)$ for $j = 1, 2, \ldots, k - 1$, and hence that M^k is partially separated. The conclusion of the theorem now follows from Theorem 4.

It is interesting to recall that if M is limit-circle then all powers M^k of M are known to be partially separated (see [18]). Hence the powers M^k can fail to be partially separated only in the case when M is limit-point but its coefficients do not satisfy the Levinson criterion (1.2). Examples of such coefficients are given in [6].

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